International Journal of Theoretical Physics, Vol. 45, No. 9, September 2006 (© 2006)
DOI: 10.1007/s10773-006-9142-z

Lie Symmetries for Hamiltonian Systems Methodological Approach

Rodica Cimpoiasu[1,2] and Radu Constantinescu[1]

Received October 19, 2005; Accepted March 20, 2006
Published Online: May 10, 2006

This paper proposes an algorithm for the Lie symmetries investigation in the case of a 2D Hamiltonian system. General Lie operators are deduced firstly and, in the the next step, the associated Lie invariants are derived. The 2D Yang-Mills mechanical model is chosen as a test model for this method.

KEY WORDS: Lie symmetries; invariants; Yang-Mills mechanical model.

PACS: 05.45.-a; 02.30.Ik.

1. INTRODUCTION

Nowadays, dynamical systems are intensively studied and play an important role in the sphere of basic theoretical researches from mathematics and physics, and in the application spheres of these related branches of science.

Chaos is characteristic for many nonlinear dynamical systems with finite or infinite degrees of freedom. In nature, chaotic behavior more is a rule than an exception. The investigation methods of chaotic dynamics are studied in Lichtenberg and Lieberman (1994). To prove that some of the dynamical systems are chaotic (nonintegrable systems), a study of the periodical orbits of the system can be done, case in which this analysis is possible, through numerical methods. A special characteristic of the classes of periodical solutions is, in the case of chaotic systems, their instability at small perturbations of the initially chosen conditions for the system. A frequently used method to observe the chaotic evolution of the system, at various energy values, uses of the Poincaré surface of section into which the intersections of the trajectories of the system with the phase plane (r,p) appear.

Opposite to the chaotic dynamical systems are the integrable ones, which present a regular behavior. Through the perturbation of the Hamiltonian from an

[1] University of Craiova, Craiova, Romania.
[2] To whom correspondence should be addressed at University of Craiova, 13 A.I.Cuza, 200585 Craiova, Romania; e-mail: rodicimp@central.ucv.ro.

integrable system, we obtain a chaotic system. In Ciraolo *et al.* (2003), a method of the chaos' control by adding to the Hamiltonian a control term, which has an order smaller than the perturbation, and leading to a quasi-integrable system with a more regular dynamics is proposed. In Oloumi and Teychenné (1999), Sirko and Koch (2002), a local control of chaos, respectively a control using the variations of the external field are performed.

In a tight connection with integrability is the problem of isolating the constants of motion for a given physical system. The determination of the invariants for autonomous or non-autonomous Hamiltonian integrable systems can be done by using direct or indirect methods. For example in Struckmeier and Riedel (2000), an invariant for a three dimensional Hamiltonian systems of N particles with the potential depending explicitly on time is derived by a direct method. The invariant is found to contain a time-dependent function, embodying a solution of a third-order differential equation.. In the case of a 2-dimensional autonomous system, where the Hamiltonian does not depend explicitly on time, a first constant quantity is immediately found: the Hamiltonian itself that represents the system's total energy. Therefore, the only problem to be solved in order to fulfill the integrability condition would be to find a second invariant.

The indirect methods for constructing the invariants consist in their determining from the symmetries found for the analyzed system. Both, the Lie symmetries which leave invariant the evolution equations of the system (leave invariant the differential equation with partial derivatives which described the physics' process) and the Noether symmetries which leave invariant the action of the system, are in the attention of researchers. In Struckmeier and Riedel (2002), are pointed out the Noether and Lie symmetries and the associated invariants too, for the non-autonomous Kepler system. In Ablowitz *et al.* (1980), Olshanetsky and Perelomov (1981), two main indirect approaches of integrability are used: the Painlevé analysis, respectively the Lax-pair method. Another method to generate symmetries for a dynamical system is based on the notion of a recursion operator. This algorithm generates infinite hierarchies of symmetries as well as infinite families of conservation laws depending on higher order derivatives of variables. In the recent years, were also investigated another type of symmetries for various models. A special type of nonlocal symmetries are investigated in Gandarias (2001), Geronimi *et al.* (2001).

The way followed in this paper is a more direct one. It consists in the check of the invariants of the system starting from its symmetries. The objective of this paper consists in presenting a possible algorithmic approach to the integrability problem of the Hamiltonian dynamical systems. In order to avoid nonsignificant mathematical complications, we will restrict ourselves to 2-dimmensional dynamical systems with polynomial interaction. Moreover, a particular case, arising from Quantum field theory and having interesting (regular or chaotic) behaviors, will be considered as a toy model: the 2D Yang-Mills mechanical model, that is a model with even order interaction.

After this introduction, we expose a general algorithm for the computation of Lie symmetries and associated constants of motion for a bidimensional general Hamiltonian system which does not depend explicitly on time. In order to capture the full invariance properties of the analyzed system, we search for Lie symmetries, in the 4D space $(x, y, \dot{x}, \dot{y})$.By this, more general Lie symmetries than the standard ones are obtained. In the third section the periodicity and regular behavior, in some conditions, for a particular Yang-Mills model is investigated and then, by applying the algorithm from the previous section, the cases of integrability and the associated invariants are evidenced. Some remarks and conclusions will end the paper.

2. GENERAL METHODOLOGY

Let us consider a Hamiltonian system described, in a 4-dimmensions phase space, by the Hamiltonian $H(x, y, \dot{x}, \dot{y})$.We restrict ourselves to a Hamiltonian of the form:

$$H = T(\dot{x}, \dot{y}) + P(x, y) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + V(x, y, a_p) \tag{1}$$

where $\{a_p,\ p = 1 \ldots\}$ is an arbitrary set of constants.

The equations of motion have the form:

$$\ddot{x} = -\frac{\partial H}{\partial x} = -\frac{\partial V}{\partial x}; \ddot{y} = -\frac{\partial H}{\partial y} = -\frac{\partial V}{\partial y} \tag{2}$$

As in Olver (1993), the symmetry operator will have the form:

$$U = \varphi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} \tag{3}$$

with a second order extension:

$$U^{(2)} = \varphi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \varphi^t\frac{\partial}{\partial \dot{x}} + \phi^t\frac{\partial}{\partial \dot{y}} + \varphi^{2t}\frac{\partial}{\partial \ddot{x}} + \phi^{2t}\frac{\partial}{\partial \ddot{y}} \tag{4}$$

The invariant conditions of the evolution equations (2) are:

$$U^{(2)}\left[\ddot{x} + \frac{\partial V}{\partial x}\right] = \varphi^{2t} + U^{(2)}\left[\frac{\partial V}{\partial x}\right] = 0 \tag{5}$$

$$U^{(2)}\left[\ddot{y} + \frac{\partial V}{\partial y}\right] = \phi^{2t} + U^{(2)}\left[\frac{\partial V}{\partial y}\right] = 0 \tag{6}$$

We will choose for the coefficient functions from (3), the following form linear in velocities:

$$\varphi(x, y, \dot{x}, \dot{y}) = f(x, y)\dot{x} + g(x, y)\dot{y} + \mu(x, y) \tag{7}$$

$$\phi(x, y, \dot{x}, \dot{y}) = h(x, y)\dot{x} + \rho(x, y)\dot{y} + \upsilon(x, y) \tag{8}$$

With this choice we compute the expressions for φ^{2t}, ϕ^{2t}, appearing in (4), in the following way:

$$\begin{aligned}\varphi^{2t} = \frac{d^2}{dt^2}\varphi &= f_{2x}\dot{x}^3 + g_{2y}\dot{y}^3 + [g_{2x} + 2f_{xy}]\dot{y}\dot{x}^2 + [f_{2y} + 2g_{xy}]\dot{x}\dot{y}^2 + \ddot{x}[3f_x\dot{x} \\ &+ g_x\dot{y} + 2f_y\dot{y} + \mu_x] + \ddot{y}[3g_y\dot{y} + f_y\dot{x} + 2g_x\dot{x} + \mu_y] + f\,\dddot{x} + g\,\dddot{y} + \mu_{2x}\dot{x}^2 \\ &+ 2\mu_{xy}\dot{x}\dot{y} + \mu_{2y}\dot{y}^2 \end{aligned} \tag{9}$$

$$\begin{aligned}\phi^{2t} = \frac{d^2}{dt^2}\phi &= h_{2x}\dot{x}^3 + \rho_{2y}\dot{y}^3 + [\rho_{2x} + 2h_{xy}]\dot{y}\dot{x}^2 + [h_{2y} + 2\rho_{xy}]\dot{x}\dot{y}^2 + \ddot{x}[3h_x\dot{x} \\ &+ \rho_x\dot{y} + 2h_y\dot{y} + \upsilon_x] + \ddot{y}[3\rho_y\dot{y} + h_y\dot{x} + 2\rho_x\dot{x} + \upsilon_y] + h\,\dddot{x} + \rho\,\dddot{y} + \nu_{2x}\dot{x}^2 \\ &+ 2\nu_{xy}\dot{x}\dot{y} + \nu_{2y}\dot{y}^2 \end{aligned} \tag{10}$$

where $\ddot{x}$, $\ddot{y}$ are the forms (2).

Coming back with the previous expressions in the invariance conditions (5), (6) and vanishing the coefficients of various monomials of the form $\dot{x}^a\dot{y}^b$, $a, b = 0, 1, 2 \ldots$, we obtain a system S of certain number of equations with the unknown functions f, g, h, ρ, μ, ν, from (7), (8). Solving this system, we obtain solutions of the form:

$$f\big(x, y, c_i^{(f)}\big), g\big(x, y, c_j^{(g)}\big), h\big(x, y, c_k^{(h)}\big), \rho\big(x, y, c_l^{(\rho)}\big), \mu\big(x, y, c_m^{(\mu)}\big), \nu\big(x, y, c_n^{(\nu)}\big) \tag{11}$$

where a certain number of arbitrary constants $c_i^{(f)}$, $c_j^{(g)}$, $c_k^{(h)}$, $c_l^{(\rho)}$, $c_m^{(\mu)}$, $c_n^{(\nu)}$, not all independent, appear in the expression of each determined function. The number of independent Lie symmetry operators for the Hamiltonian system (1), depends on the number of independent constants from the previous set.

Let us consider that $\{U_r,\ r = \overline{1, n}\}$ represents the set of independent Lie symmetry operators for the system under consideration.

The next step of the algorithm consists in finding the invariants $\{I_r(x, y, \dot{x}, \dot{y}),\ r = \overline{1, n}\}$ associated to independent set of Lie operators which are determined in the previous step. These invariants will be solutions of the equations:

$$U_r^{(1)} I_r = 0, r = \overline{1, n} \tag{12}$$

where $U_r^{(1)}$ represent the first order extensions of U_r and have the forms:

$$U_r^{(1)} = \varphi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \varphi^t\frac{\partial}{\partial \dot{x}} + \phi^t\frac{\partial}{\partial \dot{y}} = \varphi\frac{\partial}{\partial x} + \phi\frac{\partial}{\partial y} + \dot{\varphi}\frac{\partial}{\partial \dot{x}} + \dot{\phi}\frac{\partial}{\partial \dot{y}} = 0 \tag{13}$$

The set of second invariants $\{I_r\}$, $r = \overline{1, n}$ will be determined by integrating the equations Lakshmanan and Velan (1992)

$$\begin{aligned} \frac{\partial I_r}{\partial x} &= -\dot{\varphi}, \frac{\partial I_r}{\partial \dot{x}} = \varphi \\ \frac{\partial I_r}{\partial y} &= -\dot{\phi}, \frac{\partial I_r}{\partial \dot{y}} = \phi \end{aligned} \tag{14}$$

It is important to notice that, each Lie symmetry and the associated invariant correspond to a different situation, that is to different concrete forms of the Hamiltonian function. The system S allows in addition to determine the values or the relations between the parameters a_p from the Hamiltonian (1).

3. APPLICATION: THE YANG-MILLS MECHANICAL MODEL

3.1. Periodicity and Regular Behavior

Let us consider for the Yang-Mills model, in 4 dimensional phase space, the following general form of the Hamiltonian:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{A}{2}x^2 - \frac{B}{2}y^2 + ax^2y^2 + bx^4 + dy^4 \tag{15}$$

where A, B, a, b, d are parameters.

The equations of motion have the forms:

$$\begin{aligned} \ddot{x} &= -\frac{\partial H}{\partial x} = Ax - 2axy^2 - 4bx^3 \\ \ddot{y} &= -\frac{\partial H}{\partial y} = By - 2ayx^2 - 4dy^3 \end{aligned} \tag{16}$$

If we should consider the previous system in the 4 dimensional space generated by $(x, y, \dot{x}, \dot{y})$, we could look for various classes of solutions of the system. It is not possible to give a list with all the solutions of the system, but, as a rule, a first step towards integrability is suggested by the existence of periodical solutions. To be more concrete, we will analyze in this section a particular case of (15), namely the case with:

$$H = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}x^2 + \frac{1}{2}y^2 + x^2y^2 + x^4 + y^4 \tag{17}$$

In this case classes of periodical solutions can be found by numerical investigations. Our effective studies were done by considering the following set of initial conditions: $x_0 = 0$; y_0; $\dot{x}_0 = \sqrt{2E}\cos\alpha$; $\dot{y}_0 = \sqrt{2E}\sin\alpha$, where y_0 and α are considered as parameters and E represents the energy of the system. In Fig. 1 we considered $y_0 = 0$ and various values for α. Figure 2 corresponds to the case $\alpha = 0$ and y_0 variable. In the two cases we have chosen the energy $E = 2$. In Fig. 3 the projections of the orbits of the Hamiltonian (17), in the surface of section $x = 0$ are shown for various values of the energy.

The main feature of the system consists in the simultaneous presence of regular trajectories and of stochasticity regions at high value of the energy. For the formers the trajectory strikes the Poincaré surface in some fixed points, called also *periodic points* and determines a closed *invariant curve*. The latter can be identified by the fact that the successive intersections of the trajectory with the considered surface do not come twice to the same point and densely cover the surface during long periods of time. It is interesting to notice that, by increasing the energy, regular orbits disappear and chaotic areas are extended. For the values of energy $E = 10$, $E = 4000$, one can see how periodic points could generate islands of stability and for $E = 40000$, the chaotic behavior appear. The system (17) is more stable than the another particular case ($A = B = -1$, $a = 1$, $b = d = 0$) for (15), considered in Cimpoiasu (2005), because in the first case, chaotic regions appear at much higher value of the energy.

3.2. Integrability Cases for the 2D Yang-Mills Model

In this section we intend to show how the algorithm presented in Section 2 works. We will apply it for the Yang-Mills model given by (15) with $d = 1$ and we will derive all the integrability cases of the model.

The invariance conditions for the equations (16) have the forms:

$$\varphi[-A + 2ay^2 + 12bx^2] + \phi[4axy] + \varphi^{2t} = 0 \tag{18}$$

$$\varphi[4axy] + \phi[-B + 2ax^2 + 12y^2] + \phi^{2t} = 0 \tag{19}$$

By introducing in (18), (19) the chosen forms (7), (8) the coefficient function (9), (10) in which we substitute the particular equations (16) and their derivatives in time, and by vanishing the coefficients of various monomials of the form $\dot{x}^a\dot{y}^b$, $a, b = 0, 1, 2, 3$, we obtain the set of equations:

$$f_{2x} = 0, \quad g_{2y} = 0 \tag{20}$$

$$2g_{xy} + f_{2y} = 0, \quad 2f_{xy} + g_{2x} = 0 \tag{21}$$

$$h_{2x} = 0, \quad \rho_{2y} = 0 \tag{22}$$

$$2\rho_{xy} + h_{2y} = 0, \quad 2h_{xy} + \rho_{2x} = 0 \tag{23}$$

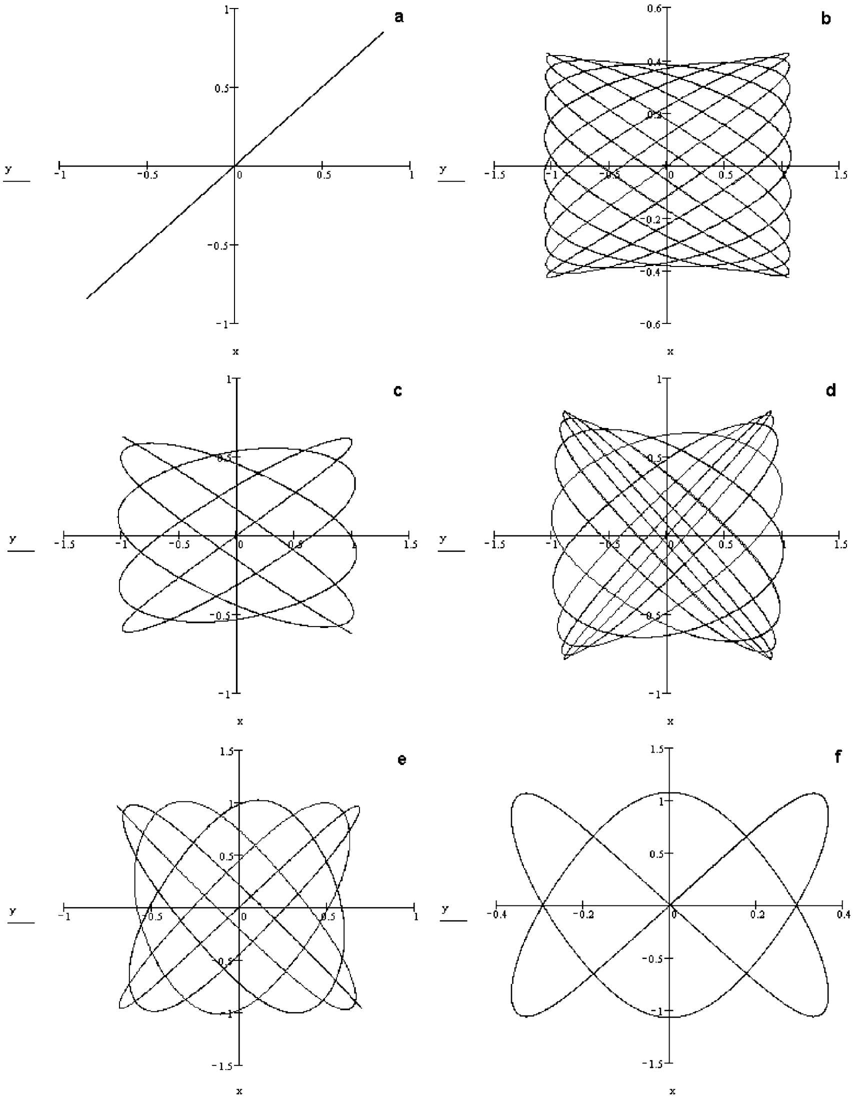

Fig. 1. Some periodical solutions for the initial conditions: $y_0 = 0$ and (a) $\alpha = \pi/4$; (b) $\alpha = 0.3$; (c) $\alpha = 0.4897$; (d) $\alpha = 0.6982$; (e) $\alpha = -1.0005$; (f) $\alpha = 1.308$.

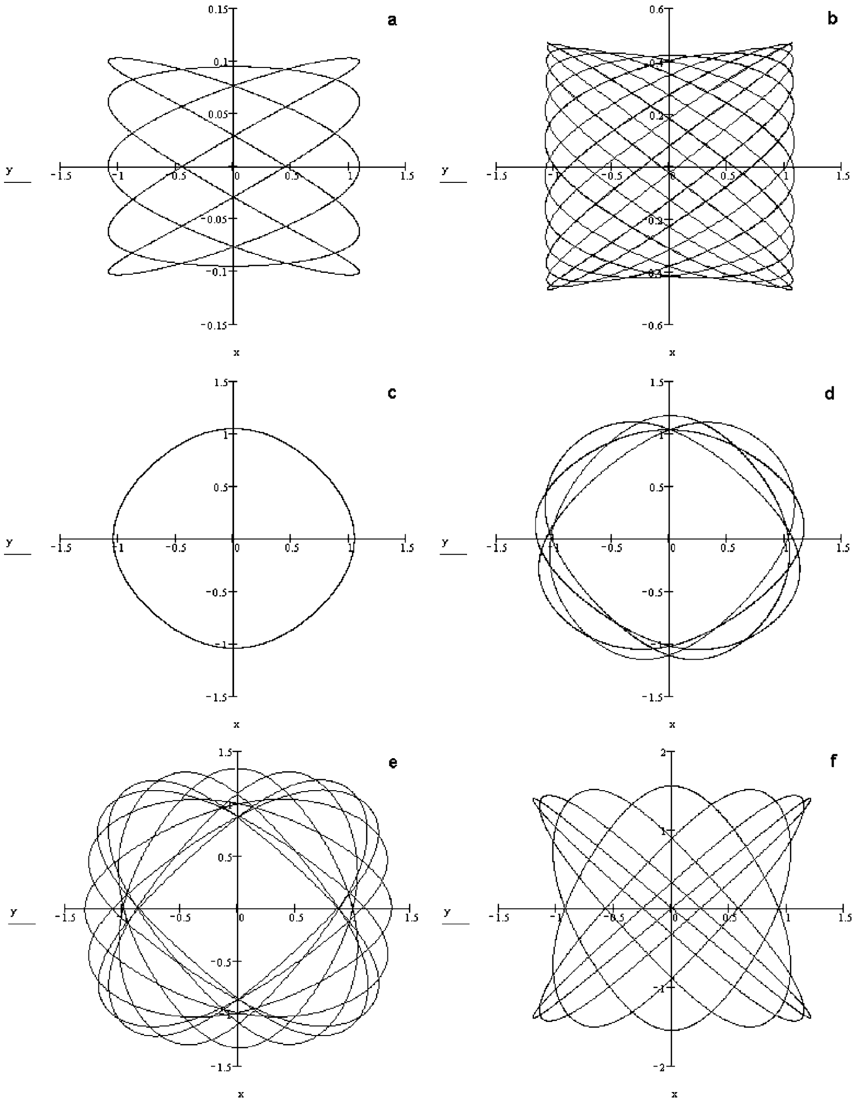

Fig. 2. Some periodical solutions for the initial conditions: $\alpha = 0$ and (a) $y_0 = 0.095$; (b) $y_0 = 0.422$; (c) $y_0 = 1.046$; (d) $y_0 = 1.170$; (e) $y_0 = 1.330$; (f) $y_0 = 1.5505$.

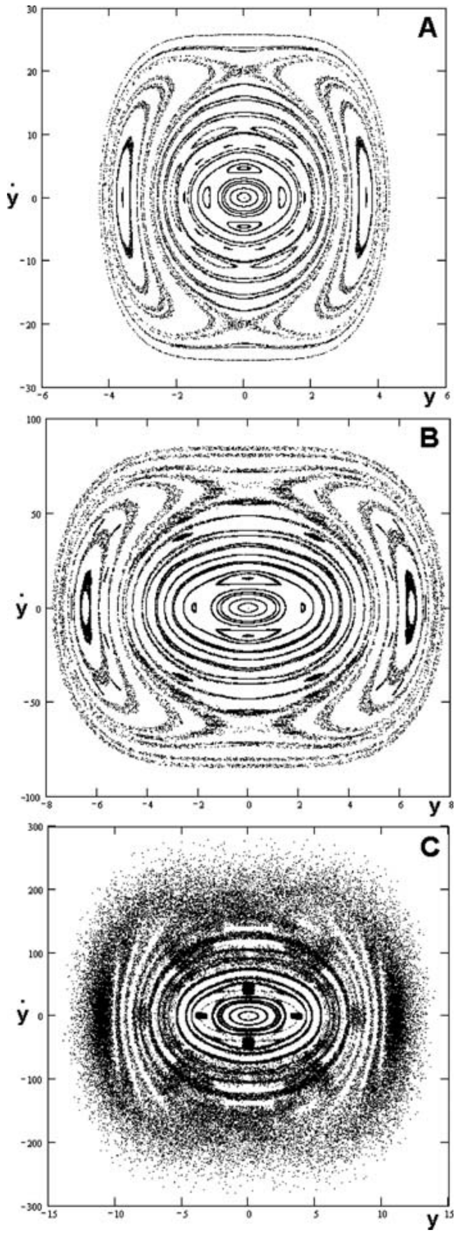

Fig. 3. The evolution of the dynamical system to chaotic behavior, in the plane $(\dot{y}, y)$,depending on its energy level: (A) $E = 10$; (B) $E = 4000$; (C) $E = 40000$.

$$\mu_{2x} = 0, \quad \mu_{2y} = 0, \quad \mu_{xy} = 0 \tag{24}$$

$$\nu_{2x} = 0, \quad \nu_{2y} = 0, \quad \nu_{xy} = 0 \tag{25}$$

$$0 = 4axy\rho + [-A + 2ay^2 + 12bx^2 + B - 2ax^2 - 12y^2]g + [Ax - 2axy^2 - 4bx^3][g_x + 2f_y] + 3[By - 2ayx^2 - 4y^3]g_y - 4axyf \tag{26}$$

$$4axyh + 3[Ax - 2axy^2 - 4bx^3]f_x + [By - 2ayx^2 - 4y^3][2g_x + f_y] - 4axyg = 0 \tag{27}$$

$$[Ax - 2axy^2 - 4bx^3]\mu_x + [By - 2ayx^2 - 4y^3]\mu_y + [-A + 2ay^2 + 12bx^2]\mu + 4axy\upsilon = 0 \tag{28}$$

$$4axyg + 3[By - 2ayx^2 - 4y^3]\rho_y + [Ax - 2axy^2 - 4bx^3][\rho_x + 2h_y] - 4axyh = 0 \tag{29}$$

$$0 = 4axyf + [-B + 2ax^2 + 12y^2 + A - 2ay^2 - 12bx^2]h + [By - 2ayx^2 - 4y^3][h_y + 2\rho_x] + 3[Ax - 2axy^2 - 4bx^3]h_x - 4axy\rho \tag{30}$$

$$[Ax - 2axy^2 - 4bx^3]\upsilon_x + [By - 2ayx^2 - 4y^3]\nu_y + [-B + 2ax^2 + 12y^2]\upsilon + 4axy\mu = 0 \tag{31}$$

The set of Equations (20)–(25) offers the solutions:

$$f(x, y) = -c_5 y^2 - \frac{1}{2}c_1 xy + c_4 x + c_7 y + c_8 \tag{32}$$

$$g(x, y) = \frac{c_1}{2}x^2 + c_5 xy + c_2 x + c_6 y + c_3 \tag{33}$$

$$h(x, y) = \frac{c_9}{2}y^2 + c_{13}xy + c_{10}y + c_{14}x + c_{11} \tag{34}$$

$$\rho(x, y) = -c_{13}x^2 - \frac{c_9}{2}xy + c_{12}y + c_{15}x + c_{16} \tag{35}$$

$$\mu(x, y) = c_{17}x + c_{18}y + c_{19} \tag{36}$$

$$\upsilon(x, y) = c_{20}x + c_{21}y + c_{22} \tag{37}$$

By the substitution of the expressions (32)–(37) into the set of equations (26)–(31), and by vanishing the coefficients of various monomials of the form $x^a y^b$, $a, b = 0, 1, 2, 3,$ we obtain a set of conditions which reduces, tacking account that parameters a and b from the hamiltonian must not be zero, to the following relations:

$$c_1 = c_4 = c_9 = c_{12} = c_{17} = c_{19} = c_{21} = c_{22} = 0 \tag{38}$$

$$c_7 = -2c_2, \quad c_{14} = c_2, \quad c_{11} = c_3, \quad c_{13} = c_5,$$

$$c_{16} - c_8 = \frac{A - B}{a} c_5, \quad c_{10} = c_6, \quad c_{15} = -2c_6 \tag{39}$$

$$c_2[3 - 4a] = 0, \quad c_2[B - 4A] = 0, \quad c_2[12b - a] = 0 \tag{40}$$

$$c_3[6b - a] = 0, \quad c_3[a - 6] = 0, \quad c_3[B - A] = 0 \tag{41}$$

$$c_5[a - 2b] = 0, \quad c_5[a - 2] = 0 \tag{42}$$

$$c_6[3b - 4a] = 0, \ c_6[a - 12] = 0, \ c_6[4B - A] = 0 \tag{43}$$

$$[B - A]c_{18} = 0, \ [a - 2]c_{18} = 0, \ [B - A]c_{20} = 0, \ [a - 2b]c_{20} = 0 \tag{44}$$

$$[a - 6b]c_{18} - 2ac_{20} = 0, [a - 6]c_{20} - 2ac_{18} = 0 \tag{45}$$

There are 4 independent cases:

Case I. $c_2 = 1$ and the remaining independent $c_i, i = 3, 5, 6$ are equal to zero.

From (39), (40), (44), it results:

$$c_7 = -2, \ \ c_{14} = 1, \ \ c_{16} = c_8 \equiv c, \ \ c_{18} = c_{20} = 0 \tag{46}$$

$$a = \frac{3}{4}, \ \ b = \frac{1}{16}, \ \ B = 4A \tag{47}$$

In this first case, the Lie symmetry operator has the form:

$$U_I = [(-2y + c)\dot{x} + x\dot{y}]\frac{\partial}{\partial x} + [x\dot{x} + c\dot{y}]\frac{\partial}{\partial y} \tag{48}$$

Case II. $c_3 = 1$ and the remaining independent $c_i, i = 2, 5, 6$ are equal to zero.

The conditions (39), (41), (44), provide the new relations:

$$c_{11} = 1, \ \ c_{16} = c_8 \equiv c, \ \ c_{18} = c_{20} = 0 \tag{49}$$

$$a = 6, \ \ b = 1, \ \ A = B \tag{50}$$

The operator (3) becomes:

$$U_{II} = [(c\dot{x} + \dot{y}]\frac{\partial}{\partial x} + [\dot{x} + c\dot{y}]\frac{\partial}{\partial y} \tag{51}$$

Case III. $c_5 = 1$ and the remaining independent $c_i, i = 2, 3, 6$ are equal to zero.

From (39), (42), (45) we obtain:

$$c_3 = 1, \ \ c_{16} = c_8 \equiv c \text{ if } A = B; \ \ c_{18} = -c_{20} \equiv c_0 \tag{52}$$

$$a = 2, \ \ b = 1 \tag{53}$$

It follows that the generator of the Lie symmetry has the expression:

$$U_{III} = [(-y^2 + c)\dot{x} + xy\dot{y} + c_0 y]\frac{\partial}{\partial x} + [(-x^2 + c)\dot{y} + xy\dot{x} - c_0 x]\frac{\partial}{\partial y} \tag{54}$$

Case IV. $c_6 = 1$ and the remaining independent c_i, $i = 2, 3, 5$ are equal to zero

From (39), (43), (44) it results:

$$c_{10} = 1, \quad c_{15} = -2, \quad c_{16} = c_8 \equiv c, \quad c_{18} = c_{20} = 0 \tag{55}$$

$$a = 12, \quad b = 16, \quad A = 4B \tag{56}$$

In the last case, the form for Lie operator is:

$$U_{IV} = [c\dot{x} + y\dot{y}]\frac{\partial}{\partial x} + [y\dot{x} + (-2x + c)\dot{y}]\frac{\partial}{\partial y} \tag{57}$$

Through the method of constructing the Lie second invariants described in the previous section, we obtain the following cases of integrability:

Case I:

$$a = \frac{3}{4}, \quad b = \frac{1}{16}, \quad B = 4A$$

$$I_I = c\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + y^4 + \frac{1}{16}x^4 + \frac{3}{4}x^2y^2 - \frac{A}{2}x^2 - 2Ay^2\right] - y\dot{x}^2 + x\dot{x}\dot{y}$$
$$+ \frac{1}{2}x^2y^3 + \frac{1}{4}yx^4 - Ax^2y \tag{58}$$

$$H_I = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{A}{2}x^2 - 2Ay^2 + \frac{3}{4}x^2y^2 + \frac{1}{16}x^4 + y^4$$

Case II:

$$a = 6, \quad b = 1, \quad A = B$$

$$I_{II} = c\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + x^4 + y^4 + 6x^2y^2 + \frac{A}{2}x^2 - \frac{A}{2}y^2\right] + \dot{x}\dot{y}$$
$$+ 4x^3y + 4xy^3 - Axy \tag{59}$$

$$H_{II} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{A}{2}x^2 - \frac{A}{2}y^2 + 6x^2y^2 + x^4 + y^4$$

Case III:

$$a = 2, \quad b = 1, \quad A = B$$

$$I_{III} = c\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + x^4 + y^4 + 2x^2y^2 - \frac{A}{2}x^2 - \frac{A}{2}y^2\right] + c_o[y\dot{x} - x\dot{y}]$$
$$- \frac{1}{2}[y\dot{x} - x\dot{y}]^2 \approx c\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + x^4 + y^4 + 2x^2y^2 - \frac{A}{2}x^2 - \frac{A}{2}y^2\right]$$
$$+ c_o[y\dot{x} - x\dot{y}] \tag{60}$$

$$H_{III} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{A}{2}x^2 - \frac{A}{2}y^2 + 2x^2y^2 + x^4 + y^4$$

Case IV:

$$a = 12, \quad b = 16, \quad A = 4B$$

$$I_{IV} = c\left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + 16x^4 + y^4 - 2Bx^2 - \frac{B}{2}y^2\right] - x\dot{y}^2 + y\dot{x}\dot{y} + 8y^2x^3 + 4xy^4 - Bxy^2 \tag{61}$$

$$H_{IV} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - 2Bx^2 - \frac{B}{2}y^2 + 12x^2y^2 + 16x^4 + y^4$$

4. CONCLUSIONS

The paper intended to give a possible way of studying the symmetries of a non-linear dynamical system and how they could be used in the direct check of its integrability. A direct method for the search of the invariants was proposed in Hietarinta (1987), where various types of invariants, linear, quartic or cubic in velocities, were considered. In Cimpoiasu *et al.* (2005), we applied this approach for two mechanical models with polynomial potentials: Yang-Mills and Hénon-Heiles systems. In this paper we followed an alternative way of determining the invariants: the Lie symmetry operators of the system are firstly obtained and the associated invariants comes from them in the next step. The advantage of the algorithm we proposed now consists in the fact that no initial assumptions on the form of the invariants are necessary, they arising in a natural way. As an application of the algorithm, the Yang-Mills system was investigated in the previous section. It is important to remark that, if we impose here $c = 0$ and $c_o = 1$, we recover all the cases of integrability and the associated invariants obtained in Cimpoiasu *et al.* (2005). Only the first three cases were listed in Kasperczuk (1994), using a different algorithm than the one referred to in this paper. At the metodological level, our approach offers a direct connection between symmetries of the non-linear systems and their invariances. Also, the "traces of integrability" for a nonintegrable Yang-Mills case was pointed out by numerical analysis. Two families of periodical solutions are respectively obtained for two set of initial conditions. It is important to note that, in the Poincaré surface of section $x = 0$, at high energy, ones of the regular orbits disappear and chaotic areas appear.

REFERENCES

Ablowitz, M. J., Ramani, A., and Segur, H. A. (1980). A connection between nonlinear evolution equations and ordinary differential equations of P-type. *Journal of Mathematical Physics* **21**, 715.

Cimpoiasu, R. (2005). Integrability features for the non-abelian gauge field. *Romanian Reports in Physics* **57**(2), 167.

Cimpoiasu, R., Constantinescu, R., and Cimpoiasu, V. (2005). Integrability of dynamical systems with polynomial Hamiltonians. *Romanian Journal of Physics* **50**(3–4), 317.

Ciraolo, G., Vittot, M., and Pettini, M. (2003). Control of chaos in Hamiltonian systems. *Reviews of Mathematical Physics* **8**.

Gandarias, M. L. (2001). New symmetries for a model of fast fast diffusion. *Physics Letters A* **286**, 153.

Geronimi, C., Feix, M., and Leach, P. (2001). Exponential nonlocal symmetries and nonnormal reduction of order. *Journal of Physics A: Mathematical and General* **34**, 10109.

Hietarinta, J. (1987). Direct methods for the search of the second invariant. *Physics Reports* **147**, 101.

Kasperczuk, S. (1994). Integrability of the Yang-Mills Hamiltonian system. *Celestial Mechanics and Dynamical Astronomy* **58**, 387.

Lakshmanan, M. (1992). Direct integration of generalized Lie summetries of nonlinear Hamiltonian systems with two degrees of freedom: Integrability and separability, M. S. Velan. *Journal of Physics A: Mathematical and General*, 1264.

Lichtenberg, A. J. and Lieberman, M. A. (1994). Regular and haotic dynamics. *Applied Mathematical Sciences* **38**.

Oloumi, A. and Teychenné, D. (1999). Controlling Hamiltonian chaos via Gaussian curvature. *Physical Review E* **60**, R 6279.

Olshanetsky, M. A. and Perelomov, A. M. (1981). Classical integrable finite-dimensional systems related to Lie-algebras. *Physics Reports* **71**, 313.

Olver, P. J. (1993). *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 110.

Sirko, L. and Koch, P. M. (2002). Control of common resonances in bichromatically driven hydrogen atoms. *Physical Review Letters* **89**, 274101.

Struckmeier, J. J. and Riedel, C. (2000). Exact invariants for a class of three-dimensional time-dependent Classical Hamiltonians. *Physical Review Letters* **85** (18), 3830.

Struckmeier, J. and Riedel, C. (2002). Noether's theorem and Lie symmetries for time-dependent Hamilton–Lagrange systems. *Physical Review E* **66**, 066605.